Quantum cryptanalysis of ECC[1]

Gustavo Banegas[2]

February 22, 2021

[1] Joint work with Daniel J. Bernstein, Iggy Von Hoof and Tanja Lange to be presented at CHES2021

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Outline

Introduction

ECC Operations

Quantum Computation

Quantum Algorithms
- Shor's algorithm

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Algorithms for Quantum Computation: Discrete Logarithms and Factoring

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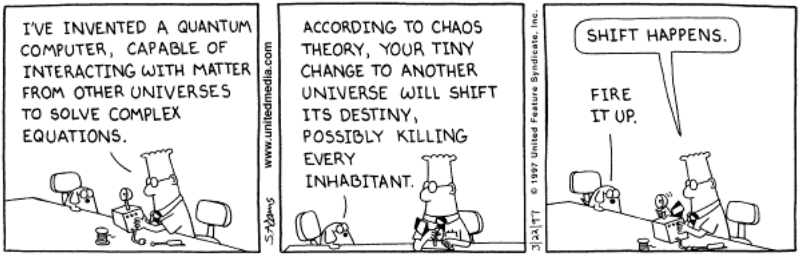
Abstract

A computer is generally considered to be a universal computational device; i.e., it is believed able to simulate any physical computational device with a cost in computation time of at most a polynomial factor. It is not clear whether this is still true when quantum mechanics is taken into consideration. Several researchers, starting with David Deutsch, have developed models for quantum mechanical computers and have investigated their computational properties. This paper gives Las Vegas algorithms for finding discrete logarithms and factoring integers on a quantum computer that take a number of steps which is polynomial in the input size, e.g., the number of digits of the integer to be factored. These two problems are generally considered hard on a classical computer and have been used as the basis of several proposed cryptosystems. (We

[1, 2]. Although he did not ask whether quantum mechanics conferred extra power to computation, he did show that a Turing machine could be simulated by the reversible unitary evolution of a quantum process, which is a necessary prerequisite for quantum computation. Deutsch [9, 10] was the first to give an explicit model of quantum computation. He defined both quantum Turing machines and quantum circuits and investigated some of their properties.

The next part of this paper discusses how quantum computation relates to classical complexity classes. We will thus first give a brief intuitive discussion of complexity classes for those readers who do not have this background. There are generally two resources which limit the ability of computers to solve large problems: time and space (i.e., memory). The field of analysis of algorithms considers the asymptotic demands that algorithms make for these resources as a function of the problem size. Theoretical

In other words..



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Introduction to Binary ECC

Basic overview

- Binary elliptic curves are elliptic curves defined over a binary field $\mathbb{F}_{2^n}$;

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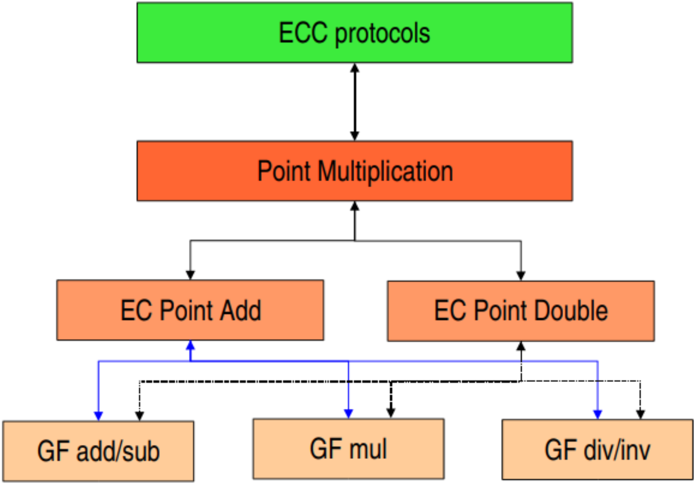
- All computations are done mod $m(z)$.

Introduction to Binary ECC

Basic overview of operations



Introduction to Binary ECC

Hardness of ECC

- Alice and Bob agrees in the same point P over a curve;

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Hardness of ECC

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- Alice selects a secret integer α and Bob selects an integer β;
- Then, they calculate and tell each other $P_\alpha = [\alpha]P$ and $P_\beta = [\beta]P$;
- Finally, they calculate their shared point $P_{\alpha\beta} = [\alpha \cdot \beta]P = [\alpha]P_\beta = [\beta]P_\alpha$.

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Quantum Computation - qubits

Qubit vs Classical bit

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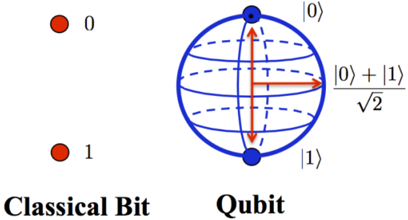
Qubit vs Classical bit



$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\alpha\,|0\rangle + \beta\,|1\rangle\,,$$

$$|\alpha|^2 + |\beta|^2 = 1$$

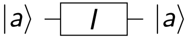
.

Measure quantum state

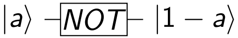
Measuring collapses the state.

Quantum gates

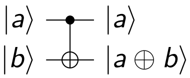
Identity gate:



NOT gate:

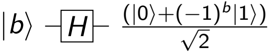


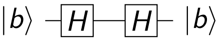
CNOT gate:



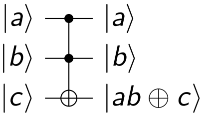
Hadamard Gate:

- $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$





Toffoli gate:



n-Qubit system

Definition

$|\psi\rangle \in \mathbb{C}^2$ such that $|| \, |\psi\rangle \, || = 1$,

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$$

where

$$\sum_{x \in \{0,1\}^n} |\alpha_x|^2 = 1.$$

Example 2-qubit system

- 4 basis states: $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$,$|1\rangle \otimes |0\rangle$, $|1\rangle \otimes |1\rangle$.
- It is common to use just: $|0\rangle\,|1\rangle$,$|10\rangle$

Quantum computation and reversibility

Reversibility

Quantum evolution is unitary (or any operation that changes the state needs to be unitary);
Unitary means:

$$UU^\dagger = U^\dagger U = I$$

Quantum computation and reversibility

Reversibility

A unitary transformation taking basis states to basis states must be a permutation.

if $U|x\rangle = |u\rangle$ and $U|y\rangle = |u\rangle$, then $|x\rangle = U^{-1}|u\rangle = |y\rangle$.

Therefore quantum mechanics imposes the constraint that classically it must be reversible computation.

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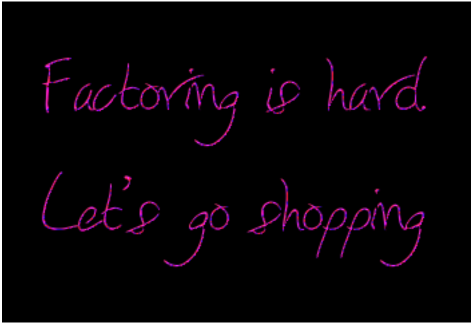
Factoring prime numbers

Factoring Integers with Shor's algorithm



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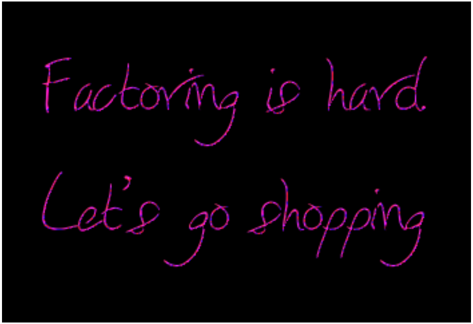

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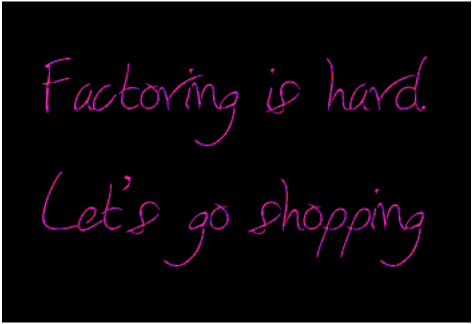

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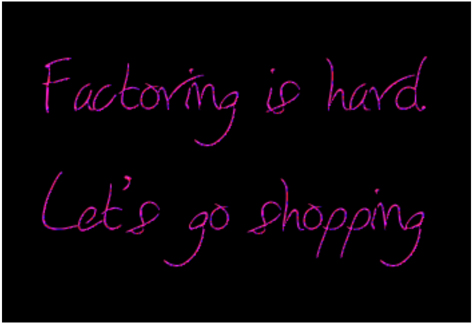

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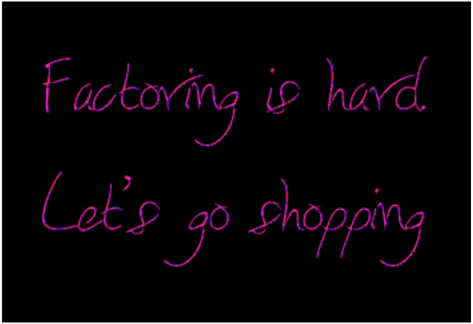

- Develop by Peter Shor in 1994;
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- It breaks RSA, ECDSA and DSA;
- How many qubits and gates do we need to run Shor's algorithm?

Shor's algorithm

In summary Shor's algorithm has two parts:

- A reduction of the factoring problem to the problem of **order-finding**, which can be done on a classical computer;

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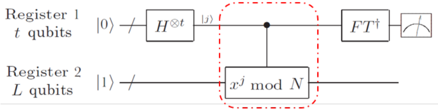
- A reduction of the factoring problem to the problem of **order-finding**, which can be done on a classical computer;
- A quantum algorithm to solve the **order-finding** problem.

Shor's algorithm

A toy example can be when we have $N = 15$. Let's see how Shor's algorithm works:

1. Select an arbitrary number, such as $a = 2$ (< 15)
2. $gcd(a, N) = gcd(2, 15) = 1$
3. Find the period of function $f(x) = a^x \mod N$, which satisfies $f(x + r) = f(x)$;
4. Get $r = 4$ through the circuit below;
5. $gcd(a^{\frac{r}{2}} + 1, N) = gcd(5, 15) = 5$;
6. $gcd(a^{\frac{r}{2}} - 1, N) = gcd(3, 15) = 5$;
7. For $N = 15$, the two decomposed prime numbers are 3 and 5.



Ressource Estimation

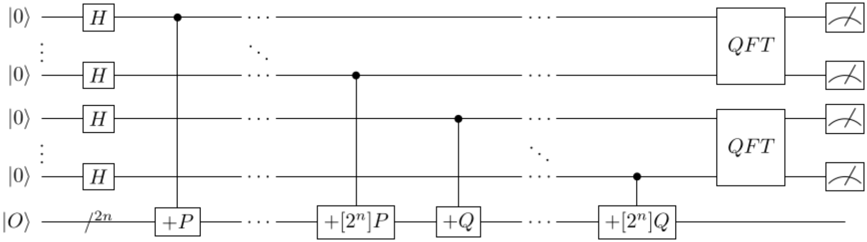
Break RSA (Integer Factoring)

From Gidney & Ekerå(2019)[3] uses "$3n + 0.002nlg(n)$ logical qubits, $0.3n^3 + 0.0005n^3lg(n)$ Toffolis, and $500n^2 + n^2lg(n)$ measurement depth to factor n-bit RSA integers"

RSA Bits	Qubits	Toffoli + T Gates (billions)
1024	3092	0.4
2048	6189	2.7
3072	9287	9.9

[3]Craig Gidney and Martin Ekerå. How to factor 2048 bit RSA integers in 8 hours using 20 million noisy qubits. arXiv preprint quant-ph/1904.09749, 2019. https://arxiv.org/abs/1905.09749

Shor's circuit for finding elliptic curve discrete logarithm



Shor's circuit for finding elliptic curve discrete logarithm

- Implementation (Quantumly) of Inversion using GCD and FLT (Fermat's little theorem);

[4] Iggy van Hoof. Space-efficient quantum multiplication of polynomials for binaryfinite fields with sub-quadratic Toffoli gate count.Quantum Information & Computation, pages 721–735, 2020.https://arxiv.org/abs/1910.02849.

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- Implementation of quantum Point addition and Point "doubling";
- Present the a quantum version of "window" addition;
- Q# implementation of Karatsuba and other functions.

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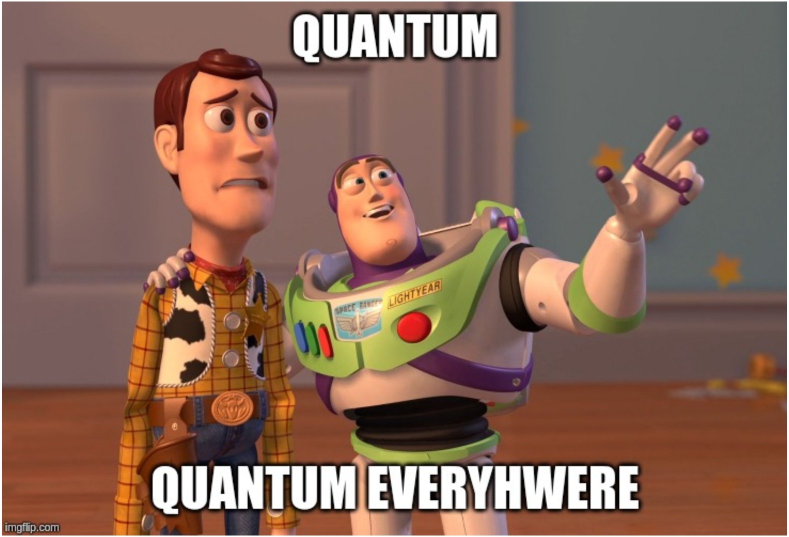
Ressource Estimation

Break Binary ECC (DLP)

From Banegas, Bernstein, von Hoof and Lange(2021)[5] we have that for breaking binary ECC we have $7n + \lfloor \log(n) \rfloor + 9$ qubits, $48n^3 + 8n^{\log(3)+1} + 352n^2 \log(n) + 512n^2 + O(n^{\log(3)})$ Toffoli gates and $O(n^3)$ CNOT gates (More details in the presentation at CHES2021).

		Single step			Total
n	qubits	TOF gates	CNOT gates	depth upper bound	TOF gates
163	1,157	893,585	827,379	1,262,035	293,095,880
233	1,647	1,669,299	1,614,947	2,405,889	781,231,932
283	1,998	2,427,369	2,358,734	3,503,510	1,378,745,592
571	4,015	8,987,401	9,080,190	13,237,682	10,281,586,744

[5] Banegas, G., Bernstein, D. J., van Hoof, I., Lange, T. Concrete quantum cryptanalysis of binary elliptic curves. IACR Transactions on Cryptographic Hardware and Embedded Systems, 2021(1)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Other Quantum algorithms

- Simon's Algorithm (QFT);
- Ambaini's Algorithm (Element disticness);
- Claw finding Algorithm;
- Kuperberg's Algorithm (dihedral hidden subgroup problem);

Questions

Thank you for your attention.
Questions?
gustavo@cryptme.in